DAMPING FACTOR TO THE PRANDTL EQUATION FOR THE TRANSITION PORTION OF A TURBULENT BOUNDARY LAYER

L. G. Loitsyanskii

UDC 532.526.4

The semiempirical theory of interaction of molecular and turbulent viscosities [1] is used to calculate the damping factor to the Prandtl equation and the average velocity distribution over the cross section of the transition portion of a turbulent boundary layer.

1. Derivation of General Equations of the Damping Factor for Quasiparallel Shear Flow in the Absence of Longitudinal Pressure Transfer. Digressing in a number of cases from the substantial role of "heredity" in a turbulent boundary layer [2], in what follows we base the discussion on the classical Boussinesq hypothesis, according to which the tangential shear stress τ, equal to the sum of the ordinary viscous laminar τ_l and the turbulent τ_t is determined by the equation

$$\tau = \tau_l + \tau_t = \mu du/dy - \rho \langle u'v' \rangle = (\mu + \mu_t)\, du/dy = \mu du/dy \left(1 + \frac{\nu_t}{\nu}\right). \tag{1}$$

For τ_t, μ_t, and ν_t we retain the usual Prandtl equations:

$$\tau_t = \rho l^2 \left(\frac{du}{dy}\right)^2, \quad \mu_t = \rho l^2 \frac{du}{dy}, \quad \nu_t = l^2 \frac{du}{dy}, \tag{2}$$

implying, however, under the "displacement method" not the usual, Prandtl mixing length l_P, equal in the transition region to $l_P = \varkappa y$, but a more general one, taking into account the effect of molecular viscosity on molar, turbulent transport. Thus, we obtain

$$\tau_t = \rho l^2 \left(\frac{du}{dy}\right)^2 = \left(\frac{l}{l_P}\right)^2 \rho l_P^2 \left(\frac{du}{dy}\right)^2 = D\tau_P, \tag{3}$$

where

$$\tau_P = \rho l_P^2 \left(\frac{du}{dy}\right)^2 = \rho \varkappa^2 y^2 \left(\frac{du}{dy}\right)^2, \tag{4}$$

and

$$D = (l/l_P)^2 \tag{5}$$

is a correction factor to the Prandtl stress of turbulent shear, called "damping factor," and defines it as a square of the ratio of the real one, taking into account the effect of molecular viscosity of the "mixing length" l to the Prandtl l_P, where this effect is neglected.

Two basic variables were introduced in [1], describing the motion in the transition region of a turbulent boundary layer:

1) the local Reynolds number

$$R = \frac{l_P^2\, du/dy}{\nu} = \frac{\varkappa^2 y^2 du/dy}{\nu}, \tag{6}$$

2) the transition function from this number

$$f(R) = 1 + \frac{\nu_t}{\nu}. \tag{7}$$

The introduction of these two concepts can validate that the Prandtl mixing length l_P is a characteristic local "linear scale," while its derivative at the local velocity gradient $l_P du/dy$ is a characteristic "velocity scale" at the given point of the normal to the solid

M. I. Kalinin Leningrad Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 45, No. 6, pp. 924-932, December, 1983. Original article submitted December 21, 1982.

0022-0841/83/4506-1381$07.50

wall cross section of the turbulent boundary layer. As to the "transition function" f(R), it reflects the transition from laminar motion directly near the solid wall, when $\nu_t = 0$, and, consequently,

$$f(R) = 1 \quad \text{at} \quad R \to 0, \tag{8}$$

to purely turbulent motion, for which there is a simple superposition of laminar and turbulent shear far from the solid wall (in the absence of their interaction), so that

$$f(R) = 1 + \frac{\nu_{tP}}{\nu} = 1 + \frac{l_P^2\, du/dy}{\nu} = 1 + R \quad \text{at} \quad R \to \infty. \tag{9}$$

From definition (7) of the transition function f(R) and the local Reynolds number R (6), if directly follows that

$$f - 1 = \frac{\nu_t}{\nu} = \frac{l^2 du/dy}{\nu} = \left(\frac{l}{l_P}\right)^2 \frac{l_P^2\, du/dy}{\nu} = DR,$$

which leads to the following general equation for the dependence of the damping factor D(R) on the transition function f(R):

$$D(R) = \frac{f(R) - 1}{R}. \tag{10}$$

To relate the local Reynolds number R with the ordinate y of the given point of the boundary layer cross section, we stay on the simplest case of quasirectilinear purely shear motion, when the pressure is constant over the whole flow region, and the total shear stress τ is also constant and equal to the friction stress at the wall τ_w. In the transition region this motion differs little from the similar motion in the longitudinal flow of the boundary layer on the film.

According to (1) and (7), we have

$$\tau = \mu \frac{du}{dy} f(R) = \tau_w. \tag{11}$$

Comparing this equality with the definition of the local Reynolds number (6), and eliminating du/dy, we obtain

$$Rf(R) = \frac{\varkappa^2 y^2 (\tau_w/\rho)}{\nu^2}. \tag{12}$$

We transform now to the "universal" coordinates

$$\eta = y v_*/\nu, \quad \varphi = u/v_*; \quad v_* = \sqrt{\tau_w/\rho}. \tag{13}$$

From equality (12) then follows the unknown relation between R and η:

$$\eta = \frac{1}{\varkappa} \sqrt{Rf(R)}. \tag{14}$$

This equation, basic for what follows, leads, together with (10), to a parametric determination of the damping factor as a function of η.

2. Average Velocity Distribution over the Cross Section of the Transition Portion of the Turbulent Boundary Layer. The equality

$$\tau = \mu du/dy + \rho l^2 (du/dy)^2 = \mu du/dy + \rho \varkappa^2 y^2 D(y) (du/dy)^2, \tag{15}$$

directly following from (1)-(3), acquires the following form in the transition to the universal variables η and φ (13)

$$1 = d\varphi/d\eta + \varkappa^2 \eta^2 D(\eta) (d\varphi/d\eta)^2. \tag{16}$$

Solving for $d\varphi/d\eta$, we obtain

$$\frac{d\varphi}{d\eta} = \frac{2}{1 + \sqrt{1 + 4\varkappa^2 \eta^2 D(\eta)}}, \tag{17}$$

$$\varphi = u/v_* = \int_0^\eta \frac{2 d\eta}{1 + \sqrt{1 + 4\varkappa^2 \eta^2 D(\eta)}}. \tag{18}$$

Denoting by η_1 the universal ordinate of the cross section point corresponding approximately to the outer boundary of the transition region, or in other words, to the beginning of the logarithmic portion of the velocity profile, we define η_1 as the root of the approximate equation (η_1, according to experimental data, is on the order of 25-30, and $D(\eta_1) = 1$):

$$\frac{2}{1+\sqrt{1+4\varkappa^2\eta_1^2 D(\eta_1)}} = \frac{1}{\varkappa\eta_1}. \tag{19}$$

We rewrite (18) in the form

$$\varphi(\eta) = \int_0^{\eta_1} \frac{2d\eta}{1+\sqrt{1+4\varkappa^2\eta^2 D(\eta)}} + \int_{\eta_1}^{\eta} \frac{d\eta}{\varkappa\eta} = \int_0^{\eta_1} \frac{2d\eta}{1+\sqrt{1+4\varkappa^2\eta^2 D(\eta)}} - \frac{1}{\varkappa}\ln\eta_1 + \frac{1}{\varkappa}\ln\eta. \tag{20}$$

Comparing (20) with the logarithmic velocity profile in the turbulent flow "core"

$$\varphi = u/v_* = A\lg\eta + B, \tag{21}$$

we determine the constants A and B:

$$A = \frac{2.303}{\varkappa}, \quad B = \int_0^{\eta_1} \frac{2d\eta}{1+\sqrt{1+4\varkappa^2\eta^2 D(\eta)}} - \frac{2.303}{\varkappa}\lg\eta_1, \tag{22}$$

where the numerical factor 2.303 is the module of decimal logarithms.

3. Determination of the Transition Function. Only two boundary values can serve to establish the form of the transition function: (8) and (9). This is, ultimately, insuffcient for characterizing its role in motion processes in the transition region.

Based on simple relaxation considerations, the following transition function was introduced in [1]

$$f(R) = 1 + R(1 - e^{-\gamma R}), \tag{23}$$

satisfying the boundary conditions mentioned above and, besides, containing for small R the asymptotic

$$f_0(R) \doteq 1 + \gamma R^2, \tag{24}$$

refining the boundary value (8). In the same approximation we obtain from (14)

$$R \doteq \varkappa^2\eta^2, \tag{25}$$

and, consequently, from (24),

$$f_0(\eta) \doteq 1 + \gamma\varkappa^4\eta^4. \tag{26}$$

Hence follows the equality

$$\frac{\nu_t}{\nu} \doteq \gamma\varkappa^4\eta^4, \tag{27}$$

expressing the earlier theoretically derived and stated experiments of Deissler [3] and Hanratty [4] of "fourth-order law" of decrease of the "turbulent viscosity" coefficient upon approaching the solid wall. The experimentally determined coefficient γ was equal in this law to

$$\gamma = 0.0092 \text{ (according to Deissler)}, \quad \gamma = 0.0125 \text{ (according to Hanratty)}. \tag{28}$$

Due to the smallness of this coefficient, the determination of the transition function by equality (23) becomes ineffective at large R, since the approximation of the function f(R) to its asymptote at large R:

$$f_\infty(R) \doteq 1 + R \tag{29}$$

occurs at R values exceeding the available experimental data by many orders of magnitude. Indeed, putting in (23), for example, $e^{-\gamma R_1} = 0.01$, then for $\gamma = 0.0092$ we have $R_1 = 500$, while, according to (14), for $f(R_1) = 1 + R_1$ we obtain the value $\eta_1 = 1250$, which contradicts sharply the experimentally known interval of values $\eta_1 = 25$-30.

As shown by the calculations performed in [1] of the average velocity distribution in the transition region and the dependence of the heat-transfer coefficient, the Stanton number, on the Prandtl number at large values, the equation of the transition function (23) applies successfully for small η (the external boundary region of the viscous sublayer). As

to large η, the impossibility was noted in [1] of using Eq. (23) in this region, and an approximate method was suggested, consisting of the use of an intermediate collapse with a logarithm (21), similar to that used by Von Karman for the "buffer" region. The matching of these two logarithms was carried out for the value $\eta_1 = 30$ chosen from experiment. This approximate velocity profile was also used in the basic dependence of the Stanton number on the Prandtl number.

In the present study this artificial method is eliminated; the transition function is assigned fully and uniquely in the whole interval of variation of R. For this purpose one more parameter is added to γ and $\varkappa$, taking into account the specific features of motion in the transition region. As such a parameter we take, as was also done in [1], the experimental constant η_1, whose value, near $\eta_1 = 30$, is refined in what follows.

We denote by R_1 the value of the local Reynolds number corresponding to η_1. The relation between R_1 and η_1 is determined by Eq. (14), in whose right-hand side one must put the asymptotic value (29) of the transition function for large R. From the equality thus obtained

$$\eta_1 = \frac{1}{\varkappa}\sqrt{R_1(1+R_1)} \tag{30}$$

it can be concluded that for $\varkappa = 0.4$ the values of $\eta_1 = 28.7$ and $\eta_1 = 28.8$, near the one selected in [1] $\eta_1 = 30$, correspond to $R_1 = 10.99$ and $R_1 = 11.03$, equal within high accuracy to $R_1 = 11$, which we take from the surrounding value $R = R_1$ at the external boundary of the transition region. For $\varkappa = 0.41$ we would have $\eta_1 = 28$.

Mellor [5] did not seek an analytic expression for the transition function at $0 < R < R_1$, but replaced it by the result of graphical differentiation of the experimental velocity distribution curve suggested by Laufer [6]. At the point $R_1 = 11$ Mellor placed a discontinuity of the curve in the form of a finite jump to the subsequent (for $R > R_1$) distribution of the transition function.

We note in passing that in Mellor's work, published in 1966, there is no mention of the fact that the variables used by him differ only in notation from those introduced by us six years prior to that: the local Reynolds number R and the transition function f(R). Our paper at the Xth International Congress in Applied Mechanics in Stresa (Italy) in 1960, published in the conference proceedings [7], remained, obviously, unknown to Mellor.

To obtain the required continuous expression for the transition function f(R), we first put on a discontinuous form of this function

$$f(R) = \frac{1}{2}[f_0(R) + f_\infty(R)] + \frac{1}{2}[f_\infty(R) - f_0(R)]\,\mathrm{sign}\,(R - R_1), \tag{31}$$

where by $f_0(R)$ and $f_\infty(R)$ we understand the asymptotes given by equalities (24) and (29), and the following discontinuous function was introduced

$$\mathrm{sign}\,(R - R_1) = \begin{cases} 1 & \text{for } R > R_1, \\ 0 & \text{for } R = R_1, \\ -1 & \text{for } R < R_1. \end{cases} \tag{32}$$

As easily verified, expression (31) for the transition function satisfies all boundary conditions earlier mentioned, while for $R = R_1$ there is a finite jump, behind which ($R > R_1$) it acquires the given asymptotic value (29).

Recalling one of the possible analytic definitions of the sign function*:

$$\mathrm{sign}\,(R - R_1) = \lim_{\alpha\to\infty}\frac{2}{\pi}\,\mathrm{arctg}\,[\alpha(R - R_1)], \tag{33}$$

we find the following asymptotic expression (for large α) for the transition function

$$f(R;\,\alpha) = \frac{1}{2}[f_0(R) + f_\infty(R)] + \frac{1}{2}[f_\infty(R) - f_0(R)]\,\frac{2}{\pi}\,\mathrm{arctg}\,[\alpha(R - R_1)], \tag{34}$$

or, replacing $f_0(R)$ and $f_\infty(R)$ by their values (24) and (29) and putting $R_1 = 11$:

$$f(R;\,\alpha) = 1 + \frac{1}{2}R\{1 + 0.0092R + (1 - 0.0092R)\,\frac{2}{\pi}\,\mathrm{arctg}\,[\alpha(R - 11)]\}\,. \tag{35}$$

*See, for example, G. A. Korn and T. M. Korn, Manual of Mathematics, McGraw-Hill (1968).

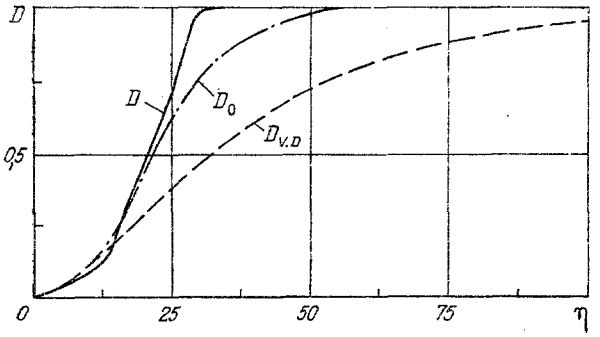

Fig. 1. Comparison of the damping factors D and D_0 with the Van Driest damping factor $D_{V.D}$.

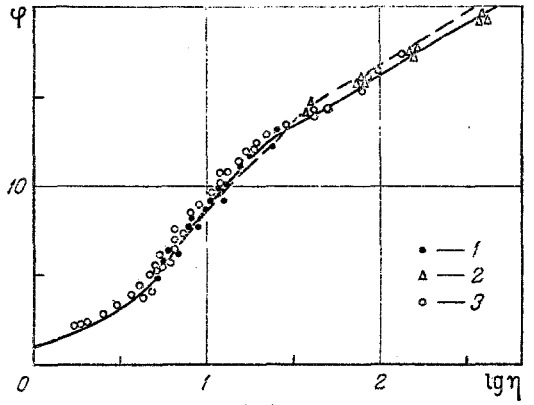

Fig. 2. Comparison of the calculated velocity curves in the transition region with experimental data: 1) [7]; 2) [9]; 3) [10].

Hence we obtain by (10) an expression for the damping factor

$$D(R;\ \alpha)=\frac{1}{2}\left\{1+0.0092R+(1-0.\ 0092R)\ \frac{2}{\pi}\operatorname{arctg}[\alpha(R-11)]\right\}. \tag{36}$$

The damping factor D(R; α) can be considered as a function playing an intermediate role in calculations. This is already observed in considering the velocity representation (18). Equally accurately the transition function f(R; α) can be written in brief form:

$$f(R;\ \alpha)=1+RD(R), \tag{37}$$

and Eq. (14) relating R and η can be written in the form

$$\eta=\frac{1}{\varkappa}\sqrt{R[1+RD(R;\ \alpha)]}. \tag{38}$$

The set of equalities (35), (36), and (38) serves as a parametric representation of the transition function f(η; α) and the damping factor D(η; α). The semiempirical theory discussed does not give explicit expressions for the transition function f(η; α) and the damping factor D(η; α) in terms of the universal ordinate η. The Van Driest empirical equation for the damping factor [8], widely used in practice,

$$D_{V.D}=[1-\exp(-\eta/A_*)]^2 \tag{39}$$

with the value of the experimental constant A_* = 26 has no relation with the semiempirical theory discussed in the present study.

As calculation show, it is quite sufficient to put α = 100, since α values higher than α = 100 provide no substantial change in the curves of the damping factor and the velocity profile. Only this α value is used in what follows, and is hence omitted in the characteristic functions f(R; α) and D(R; α).

The curves of $D(\eta)$ and $D_{V.D}(\eta)$ shown in Fig. 1 show a significant difference between them, particularly sharply expressed for large η values. If the variation region of $D(\eta)$ is restricted by values $\eta_1 \leqq 30$, the corresponding region for $D_{V.D}(\eta)$ reaches approximately $\eta_1 = 140$.

For η values near $\eta = 0$, on the other hand, both functions $D(\eta)$ and $D_{V.D}(\eta)$ coincide. Indeed, according to (10) and (24) we have

$$D(\eta) \doteq \gamma\varkappa^2\eta^2,$$

while in the same approximation

$$D_{V.D}(\eta) \doteq \frac{1}{A_*^2}\eta^2.$$

Taking $\gamma = 0.0092$ and $\varkappa = 0.4$, we obtain

$$\gamma\varkappa^2 = 0.001472,\ 1/A_*^2 = 1/26^2 \doteq 0.001479.$$

This makes it possible to conclude quite accurately that there is a connection between the constants:

$$A_* = 1/(\varkappa\sqrt{\gamma}) \tag{40}$$

with the same coincidence of the curves $D(\eta)$ and $D_{V.D}(\eta)$ near the values $\eta = 0$.

4. Comparison of Calculated Velocity Profile with Experimental Data. Using Eq. (18) and the parametric definitions (36) and (38) of the damping factor, we find by a numerical calculation the velocity distribution in the transition region of a turbulent boundary layer, shown by the solid line in Fig. 2. For comparison the dashed line shows the velocity profile calculated by Eqs. (18) and (39). Despite the sharp difference in the damping factor distributions, shown in Fig. 1, this difference is not substantial in the velocity distributions. Notable is the difference of the outer boundary $\eta = \eta_1$, equal according to the theory suggested to $\eta_1 \simeq 29$, from $\eta_1 \simeq 60$ according to Van Driest. Also notable is the systematic deviation of the dashed curve from the experimental points.

Using the calculations performed, one can determine the constants A and B in the logarithmic velocity profile (21). For the constant values $\varkappa = 0.4$, $\gamma = 0.0092$ the value B = 4.7 was determined from the numerical calculation. According to Van Driest $B_{V.D} = 5.4$, which differs substantially from the experimental 4.8-4.9. The value of A is sensitive to the choice of $\varkappa$ (A = 5.76 for $\varkappa$ = 014, and A = 5.61 for $\varkappa$ = 0.41).

The experimental points on Fig. 2 correspond to the experimental studies of Laufer [6], Wieghard [9], and Tsuji and Morikawa [10]. The observed variety in the experiments, particularly in the study of the last two authors, from which the velocity distribution was borrowed, refers to the very origin of the transition from the region (dp/dx = 0) to the diffusion region (dp/dx > 0), reflected some deviations of the experimental points from the calculated curve. Another reason for these deviations is the well-known "game of constants" ($\varkappa$ = 0.40-0.41, γ = 0.0092-0.0125, η_1 = 25-30) used in the experiments.

Under the conditions mentioned, the results of comparing calculated and experimental data, shown in Fig. 2, can be considered fully satisfactory, and the use of the semiempirical theory suggested is justified.

Along with the parametric shape of the connection between f, D, φ, and the universal ordinate η it is also useful to obtain an admittedly approximate, but simple explicit shape of this dependence. With this in mind we use the following simplified approach. Under the square root in the right-hand side of (14), we replace the transition function f(R) by its least value f(R) = 1, which leads to the dependence (25), rigorously valid only in the region of the viscous sublayer. Eliminating R from (23) and (25), we obtain the approximate equality

$$D_0(\eta) \doteq 1 - e^{-\gamma\varkappa^2\eta^2} \doteq 1 - e^{-(\eta/A_*)^2}, \tag{41}$$

which is not only not inferior in simplicity to Eq. (39), but, as can be concluded from Fig. 1 and Fig. 2 based on it, is superior to the Van Driest equation (39) in the sense of approximation to experimental data.

The hydroaerodynamics graduate student V. V. Zyabrikov participated actively in discussions of the present study, performed numerical calculations, and drew the figures.

NOTATION

τ, total shear stress; τ_p, laminar shear stress; τ_t, its turbulent portion; u, average longitudinal velocity; u' and v', fluctuating velocities; $\langle u'v' \rangle$, mean product of fluctuating velocities; μ, ν, dynamic and kinematic coefficients of molecular viscosity; μ_t, ν_t, same for the turbulent viscosity; ρ, fluid density; l and l_p, real and "Prandtl" mixing lengths; τ_p, turbulent shear stress according to Prandtl; D, D_0, and $D_{V.D}$, damping factors; $\varkappa$, Van Karman constant; f, transition function; R, local Reynolds number; $\eta = yv_*/\nu$, $\varphi = u/v_*$, universal coordinate and velocity; $v_* = (\tau_w/\rho)^{1/2}$, dynamic velocity; τ_w, shear stress at the wall; η_1, R_1, universal coordinate and local Reynolds number at the edge of the transition region; A, B, constants in the logarithmic velocity equation; γ, porportionality constant in the fourth-order law of decrease of turbulent shear upon approaching the solid wall; and A_*, a constant in the Van Driest damping factor.

LITERATURE CITED

1. Semiempirical theory of molecular and molar exchange interaction processes in turbulent fluid motion," Proc. All-Union Conf. on Theoretical and Applied Mechanics, Izd. Akad. Nauk SSSR, Moscow (1962); Mechanics of Liquids and Gases [in Russian], Nauka, Moscow (1978).
2. L. G. Loitsyanskii, "Hereditary effects in turbulent motions," Mekh. Zhidk. Gaza, No. 2, 5-19 (1982).
3. R. Deissler, "Analysis of turbulent heat transfer, mass transfer and friction in smooth tubes at high Prandtl and Schmidt numbers," NACA Rep. 1210 (1959).
4. T. Hanratty, "Study of turbulence close to a solid wall," Phys. Fluids, Suppl., 126-133 (1967).
5. G. L. Mellor, "The effects of pressure gradients on turbulent flow near a smooth wall," J. Fluid Mech., 24, 255-274 (1966).
6. J. Laufer, "The structure of turbulence in fully developed flow," NACA Rep. 1174 (1954).
7. L. G. Loitsianski, "Reciprocal action of molecular and molar transmission in turbulent motion," Proc. Xth Congr. Appl. Mech. (Stresa, Italy, 1960), Elsevier (1962), pp. 202-204.
8. E. Van Driest, "On turbulent flow near a wall," J. Aeron. Sci., 23, 1007 (1956).
9. In: Proc. Computation of Turbulent Boundary Layer, 1968 AFOSR-IFR Stanford Conference, University of Stanford, Thermosciences Division, Dep. Mech. Eng., California, USA (1969), Vol. 2, Exper. 1400, pp. 98-123.
10. J. Tsuji and G. Morikawa, "Turbulent boundary layer with pressure gradient alternating in sign," Aeron. Q., 27, No. 1, 15-28 (1976).